Approximations to the Drag Force on a Sphere Moving Slowly Through Either an Ostwald-De Waele or a Sisko Fluid

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VARIATIONAL PRINCIPLE

Motion in an isothermal continuum is described by the equations of continuity and of motion:*

$$\frac{\partial \rho}{\partial t} + (\rho u^i)_{,i} = 0 \qquad (1)$$

$$\rho\left[\frac{\partial u^i}{\partial t} + u^j u^i_{,j}\right] = -p^{,i} - \tau_{,j}^{ij} + \rho f^i \qquad (2)$$

For a complete description a relation between the viscous portion of the stress tensor τ_{ij} and the rate-of-deformation tensor d_{ij} $\left(= \frac{1}{2}[u_{i,j} + u_{j,i}]\right)$ is also required. Provided that viscoelastic phenomena are excluded, the most general relation expressing τ_{ij} as an isotropic function (*12, 14, 15*) of d_{ij} is

$$\tau_j^i = \beta\,\delta_j^i - 2\eta\, d_j^i - \eta_c\, d_k^i\, d_j^k \qquad (3)$$

in which the coefficients β, η, and η_c are functions of the invariants of d_{ij} chosen here to be

$$I = d_i^i, \quad II = d_j^i\, d_i^j, \quad III = d_j^i\, d_k^j\, d_i^k \qquad (4)$$

For an incompressible fluid $\beta = 0$ (*20*, p. 228). To date no fluids have been observed whose behavior is described by Equation (3) with $\eta_c \neq 0$ (*21*).

Many experimental data have been interpreted in terms of Equation (3) with $\beta = \eta_c = 0$ and η a function of II alone:

$$\tau_{ij} = -2\eta\, d_{ij}, \; \eta = \eta(II) \qquad (5)$$

Pawlowski (*10*; see also *3, 6, 7*) has discussed the motion of a fluid described by Equation (5) with the following restrictions:

1. The flow is time independent.
2. The inertial terms in the equation of motion are either identically zero or negligibly small as in creeping flow: $u^j u^i_{,j} = 0$.
3. The external body force f^i can be expressed in terms of a potential.
4. The fluid is incompressible ($I \equiv 0$).
5. The velocity is specified on all bounding surfaces of the system.

In this case the motion of a fluid is such that

$$J = \int_V F\, dV \qquad (6)$$

is an extremum. Here one finds

$$F = \int_0^{II} \eta(II')\; d(II') \qquad (7)$$

An expression of the type described by Equation (5) is the Sisko model (*16*) containing three parameters m_0, m_1, and n

$$\tau_{ij} = -2\,[m_0 + m_1\,(2\,II)^{(n-1)/2}]\, d_{ij} \qquad (8)$$

for which

$$F = \left[m_0 + m_1\left(\frac{2}{n+1}\right)(2\,II)^{(n-1)/2}\right] II \qquad (9)$$

For $m_0 = 0$ Equation (8) reduces to the widely used Ostwald-de Waele (or Power) model with two parameters:

$$\tau_{ij} = -2\eta\, d_{ij} = -2\,[m_1\,(2\,II)^{(n-1)/2}]\, d_{ij} \qquad (10)$$

In this case

$$F = \frac{2\eta}{n+1} II \qquad (11)$$

Equations (8) to (11) will be the basis for what follows.

APPROXIMATIONS FOR THE DRAG

For the steady state movement past a sphere of an incompressible fluid of infinite extent the macroscopic momentum balance (2) may be combined with the macroscopic mechanical energy balance (2) to obtain

$$V\, F_d = R^3 \int_0^{2\pi}\int_0^{\pi}\int_1^{\infty} -(\tau^{ij}\, u_{i,j})\, x^2 \sin\theta\, dx\, d\theta\, d\varphi \qquad (12)$$

where, since d_{ij} is a symmetric tensor

$$-\tau^{ij} u_{i,j} = 2\eta\, d^{ij} u_{i,j} = \eta\, d^{ij} u_{i,j} + \eta\, d^{ji} u_{j,i} = 2\eta\, d^{ij} d_{ij} = 2\eta\, II \qquad (13)$$

(Note that V and F_d stand for the magnitudes of the associated vectors; otherwise the sign of this equation would be incorrect, since the vectors are in opposite directions.)

At this point it is convenient to introduce the stream function ψ defined such that

$$u_r = \frac{1}{R^2 x^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{R^2 x \sin\theta}\frac{\partial \psi}{\partial x} \qquad (14)$$

In what follows it shall be assumed that the stream function may be represented by a series of the form

$$\psi = -\frac{1}{2} V R^2 x^2 \sin^2\theta \sum_{i=0}^{\infty} C_i\, G_i, \quad G_i = G_i(x) \qquad (15)$$

Zeroth Approximation for Ostwald-de Waele Fluid

If one takes

$$C_0 = 1, \; G_0 = 1 - \frac{3}{2x} + \frac{1}{2x^3}, \quad C_i = 0 \text{ for } i \neq 0 \qquad (16)$$

Equation (15) is the same as derived for the Newtonian case (*8*). After substitution of Equations (15) and (16) and rearrangement Equation (12) becomes* for a fluid described by Equation (10):

$$V\, F_d = 2\pi\, R^3\, m_1\, (3)^{\frac{(n+1)}{2}} \left(\frac{3V}{2R}\right)^{(n+1)} \int_0^{\pi}\int_1^{\infty}$$

* Comma notation stands for covariant differentiation (*9*, p. 140), and the summation convention is employed throughout. In Cartesian, orthogonal coordinates there is no distinction between the covariant and contravariant components of a tensor, and the comma denotes partial differentiation with respect to the space coordinate whose index follows. For the equations of continuity and of motion in spherical coordinates see for example (*4*, pp. 83 to 91).

* For II in spherical coordinates see (*4*, p. 91).

$$(1+H)^{\frac{(n+1)}{2}} x^{-2n} \sin\theta \, dx \, d\theta \tag{17}$$

where

$$H = H_0 = \cos^2\theta \left[1 - \frac{1}{x^2} \right]^2 + \frac{\sin^2\theta}{3x^4} - 1 \tag{18}$$

If $(1+H)^{(n+1)/2}$ is replaced by its binomial series, the operations of integration and summation may be interchanged in Equation (17) to obtain a convergent series for $(V F_d)$; (see Appendix). In terms of the drag coefficient

$$f = 2\, F_d / (\rho\, V^2\, \pi\, R^2) \tag{19}$$

the final result may be given as

$$f = 24\, X / N_{R1} \tag{20}$$

where

$$N_{R1} = 2^n \rho V^{(2-n)} R^n / m_1 \tag{21}$$

$$X = X_0 = \frac{(3)^{(3n+1)/2}}{4} \left[A + \left(\frac{n+1}{2}\right) B + \left(\frac{n+1}{2}\right) \left(\frac{n-1}{2}\right) \frac{C}{2!} + \left(\frac{n+1}{2}\right) \left(\frac{n-1}{2}\right) \left(\frac{n-3}{2}\right) \frac{D}{3!} + \left(\frac{n+1}{2}\right) \left(\frac{n-1}{2}\right) \left(\frac{n-3}{2}\right) \left(\frac{n-5}{2}\right) \frac{E}{4!} \right] \tag{22}$$

$$A = \frac{2}{2n-1}$$

$$B = \frac{10}{9}\, \frac{1}{2n+3} - \frac{4}{3}\left(\frac{1}{2n+1} + \frac{1}{2n-1} \right)$$

$$C = \frac{94}{135}\, \frac{1}{2n+7} - \frac{88}{45}\, \frac{1}{2n+5} + \frac{16}{45}\left(\frac{1}{2n+3} + \frac{3}{2n+1} + \frac{3}{2n-1} \right)$$

$$D = \frac{1}{315}\left(\frac{458}{3}\, \frac{1}{2n+11} - \frac{716}{2n+9} + \frac{3{,}332}{3}\, \frac{1}{2n+7} - \frac{320}{2n+5} - \frac{1{,}288}{2n+3} \right) + \frac{1}{105}\left(\frac{464}{2n+1} + \frac{44}{2n-1} \right)$$

$$E = \frac{1}{8505}\left(\frac{9{,}334}{3}\, \frac{1}{2n+15} - \frac{20{,}848}{2n+13} + \frac{49{,}088}{2n+11} \right) - \frac{1}{945}\left(\frac{4{,}896}{2n+9} - \frac{896}{2n+7} + \frac{256}{2n+3} \right) + \frac{256}{315}\left(\frac{1}{2n+1} + \frac{1}{2n-1} \right)$$

The factor X_0 is plotted as a function of n in Figure 1.

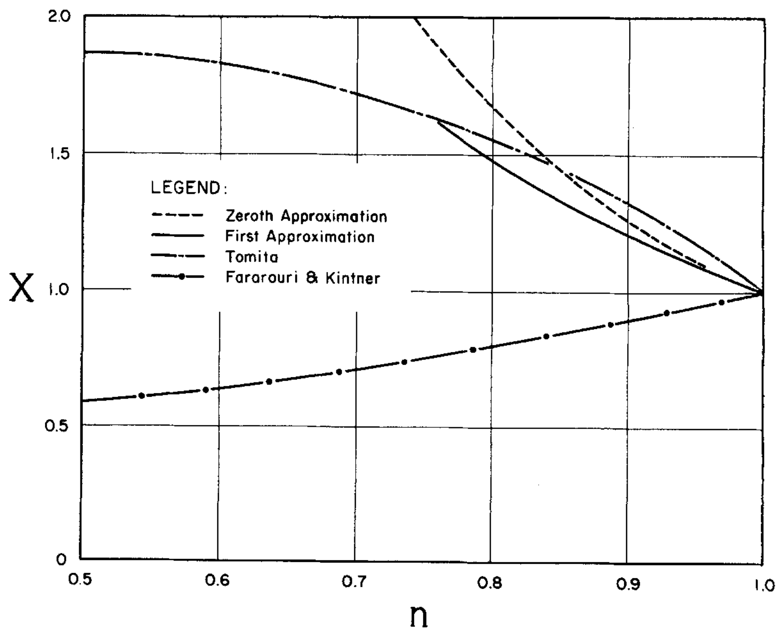

Fig. 1. The factor X of Equations (20) as a function of n.

Zeroth Approximation for Sisko Fluid

If again Equation (16) is assumed in the same manner as above, it is found that for a fluid behavior described by Equation (8)

$$f = \frac{24}{N_{R0}} + \frac{24\ X_0}{N_{R1}} \tag{23}$$

Here

$$N_{R1} = 2\, \rho\, V\, R / m_0 \tag{24}$$

and X_0 is again given by Equation (22).

The Sisko model accounts for deviations from Ostwald-de Waele behavior at higher stresses. It is inapplicable for most simple non-Newtonian fluids in the low range of stress normally associated with creeping flow, since deviations from Ostwald-de Waele behavior usually occur for stresses approaching zero.

First Approximation for Ostwald-de Waele Fluid

To obtain a first approximation to f for an Ostwald-de Waele fluid the first two terms are used in Equation (15) with C_0 and G_0 given by Equation (16). The boundary conditions on the sphere and at infinity require

$$G_1 = \frac{3}{2}\left[\frac{1}{x} - \frac{2}{x^3} + \frac{1}{x^5} \right] \tag{25}$$

leaving C_1 as an arbitrary parameter. As before the product (VF_d) is given by Equation (17) where now

$$H = H_1 = \cos^2\theta \left[a_1^2 \left(1 - \frac{12}{x^2} + \frac{46}{x^4} - \frac{60}{x^6} + \frac{25}{x^8} \right) + a_1 \left(\frac{10}{x^2} - \frac{70}{x^4} + \frac{110}{x^6} - \frac{50}{x^8} \right) + \left(\frac{25}{x^4} - \frac{50}{x^6} + \frac{25}{x^8} \right) \right] + \frac{\sin^2\theta}{3} \left[a_1^2 \left(\frac{36}{x^4} - \frac{120}{x^6} + \frac{100}{x^8} \right) + a_1 \left(-\frac{60}{x^4} + \frac{220}{x^6} - \frac{200}{x^8} \right) + \left(\frac{25}{x^4} - \frac{100}{x^6} + \frac{100}{x^8} \right) \right] - 1 \tag{26}$$

$$a_1 = 1 - C_1 \tag{27}$$

For

$$0.6376 \leq a_1 \leq 1.414 \tag{28}$$

$H^2 \leq 1$ and $(1+H)^{(n+1)/2}$ may be expanded in a convergent binomial series. As discussed in the Appendix the operations of integration and summation may be interchanged to give a convergent series for VF_d. The first three terms of this series were evaluated analytically as functions of a_1 and n. Since by Equations (11) and (13) VF_d is a constant multiple of J in Equation (6), a_1 (or C_1) was evaluated in terms of n by requiring VF_d to be a minimum. The results of this computation are given in Table 1; note that Equation (28) has been satisfied for $n > 0.76$.

TABLE 1. RESULTS OF NUMERICAL COMPUTATIONS FOR FIRST APPROXIMATION

n	a_1	x_1
1.00	1.000	1.000
0.95	0.9481	1.100
0.90	0.8855	1.212
0.891	0.8723	1.235
0.873	0.8462	1.278
0.85	—	1.34*
0.80	—	1.48*
0.763	0.6374†	1.606

* Interpolated.
† In order that the series for VF_d be convergent, $0.6376 \leq a_1 \leq 1.414$; see discussion following Equation (28).

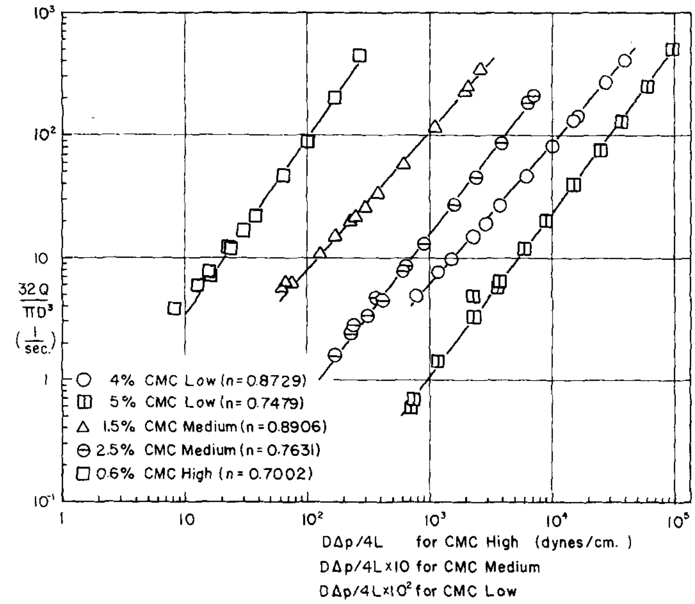

Fig. 2. Capillary-tube viscometer data (*18*) compared with Equation (31).

If the drag coefficient f is introduced, the above results may be represented by Equation (20) with $X = X_1$. The quantity X_1 as a function of n is presented both in Table 1 and in Figure 1.

PREVIOUS RESULTS

Prior attempts to analyze non-Newtonian flow past a sphere have been summarized elsewhere (*18*). More recently an analytic solution was presented for a fluid exhibiting an abrupt change in viscosity (*17*).

Three expressions for the drag force on a sphere moving slowly through an Ostwald-de Waele fluid, Equation (10), having appeared.

1. Tomita (*19*) obtained a zeroth approximation* in much the same way as Equations (20) to (22) were developed above. He assumed Equation (15) for the stream function ψ with (in the present notation)
$C_0 = 1,\ G_0 = 1 - x^{-1/n^2},$

$$C_i = 0 \text{ for } i \neq 0 \qquad (29)$$

This expression for ψ does not reduce to the correct result for $n = 1$ (*8*). No use was made of the variational principle proposed in his paper.

2. Fararoui and Kintner (*5*) suggested a highly simplified development based upon Stokes' law (*8*).

* The author wishes to thank Professor Tomita for his correspondence confirming that Tomita's Equation (29) should read

$$F(n) = \frac{n}{n^2 - n + 1} A + \ldots$$

where

$$A = \frac{2}{3}\left[1 - \frac{1}{2n}\left(1 - \frac{(n-1)^2(n+1)}{12}\right) - \frac{(n-1)(n+1)}{10n^2}\left(1 - \frac{(n-1)^2}{6} + \frac{(n-1)^4}{144}\right)\right]$$

3. A dimensionless correlation of the available experimental data has been prepared (*18*) which applied to the Ellis model* as well as to the Ostwald-de Waele model. Furthermore it is the only means for estimating the drag coefficient in a non-Newtonian fluid when inertial effects must be taken into account (for example $N_{Rl} > 1$).

* The three-parameter Ellis model may be written

$$d_{ij} = -\frac{1}{2}\left[\varphi_0 + \varphi_1 (\mathrm{II}_\tau/2)^{(\alpha-1)/2}\right]\tau_{ij} \qquad (30)$$

The results of these three developments are compared in Figure 1 with the zeroth and first approximations obtained in the above sections.

COMPARISON WITH EXPERIMENT

In the one experimental study of non-Newtonian flow past a sphere to date (*18*) the data were analyzed on the assumption that the fluids, aqueous solutions of carboxymethyl cellulose, could be described by the Ellis model, Equation (30). Two of the five solutions followed very closely the Ostwald-de Waele model, Equation (10), within the range and accuracy of the viscometer data.

In order to have some data with which to compare the developments of the second section of this paper these data were reanalyzed with the assumption that Equation (10) applied. The rheograms (plots of τ_{rz} vs. d_{rz}) formed from the capillary-viscometer data as suggested by Rabinowitsch (*11*) were fitted with Equation (10) by the method of least squares. The fluid parameters found in this manner are given in Table 2. For flow through a tube

$$\frac{32\,Q}{\pi D^3} = \frac{4n}{3n+1}\left[\frac{D\,\Delta p}{4\,L\,m_1}\right]^{1/n} \qquad (31)$$

Equation (31) is compared with the viscometer data in Figure 2; average errors are tabulated in Table 2.

By a dimensional analysis of the equation of motion for a fluid de-

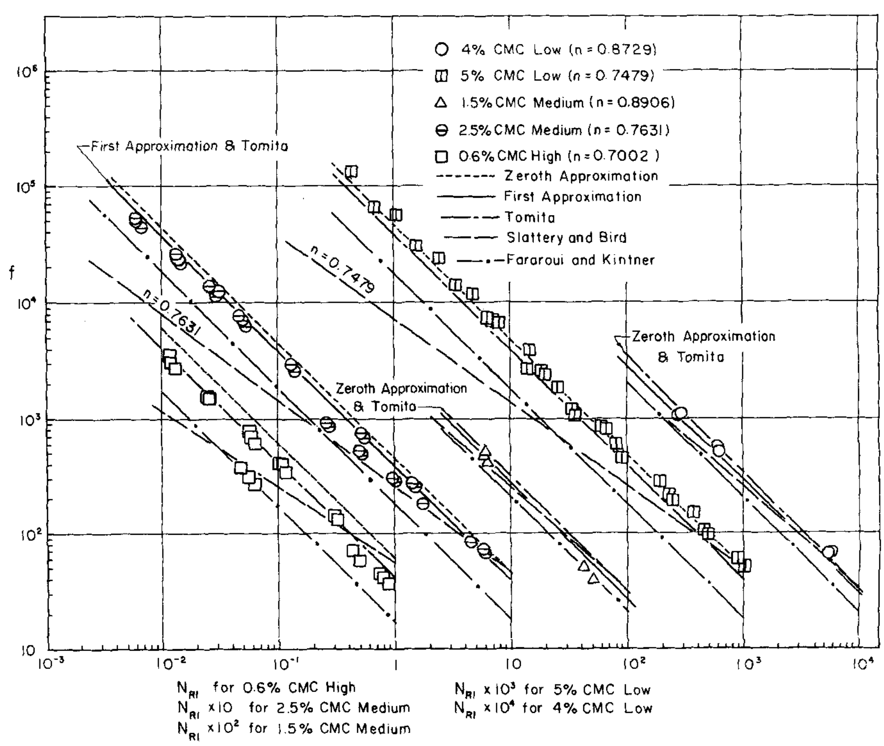

Fig. 3. Comparison of experimental values of the drag coefficient f (*18*) with various approximations.

TABLE 2. COMPARISONS WITH EXPERIMENTAL DATA

Line	Item	4% CMC low	5% CMC low	1.5% CMC medium	2.5% CMC medium	0.6% CMC high
1	n	0.873	0.748	0.891	0.763	0.700
2	m_1	2.09	9.11	1.56	12.0	3.88
3	Avg. % error in $\frac{32Q}{\pi D^3}$ with Equation (31)	1.6	2.3	5.5	4.8	8.1
4	No. of data points used to establish lines 1 to 3	22	21	24	34	26
5	**First approx.** Avg. % error in f with Equation (20) with $X = X_1$ from Figure 1	8.4		10.7	13.1	
6	**Zeroth approx.** Avg. % error in f with Equations (20) and (22)	6.6	9.5	18.3	31.2	68.85
7	Avg. % error in f with First Approx. *(19)*	6.1	20.6	22.6	17.2	23.6
8	Avg. % error in f with Slattery-Bird correlation *(18)*	20.7	58.0	19.8	36.8	45.4
9	Avg. % error in f with Fararoui-Kintner result *(5)*	36.0	63.4	19.1	45.8	49.6
10	No. of points compared in lines 5 to 9	11	42	7	46	26

scribed by Equation (10) (*4*, p. 107) it may be shown that N_{R1} is indicative of the ratio of the inertial forces to the viscous forces. It will be assumed that creeping flow (inertial forces negligibly small compared with viscous forces) exists when $N_{R1} < 1$.

Experimental values (*18*) for the drag coefficient f are plotted in Figure 3 for the five carboxymethyl cellulose solutions. Also shown are the results of the first approximation, the zeroth approximation, the corrected estimate of Tomita (*19*), the Fararoui-Kintner computation (*5*), and the dimensionless correlation of Slattery and Bird (*18*). A comparison of the average errors encountered with these developments is given in Table 2. A glance shows that the latter two are generally less accurate than the first three.

DISCUSSION

The errors accounting for the differences between calculation and experiment indicated in Table 2 fall into three categories.

1. There are two possible weaknesses in the experimental data. The fluids were not tested for viscoelastic behavior; this could be important in considering flow past a sphere. The carboxymethyl cellulose solutions for the two experiments (capillary viscometer and flow past a sphere) were prepared separately. The behaviors of the two mixes may not have been identical.

2. Equation (10) may have failed to represent adequately the fluid behavior in both experiments.

2a. It is well known that Equation (10) cannot represent the entire range of stress for a non-Newtonian fluid (*13*, p. 245). The data for the 0.6% high and 1.5% medium solutions give an indication of failure at low stresses (low values of $D\,\Delta p/4L$).

2b. Equation (10) neglects dependence upon III, although $III \neq 0$ for flow past a sphere. If the stress can be formulated in terms of a potential $\tau_{ij} = \partial\Gamma/\partial d^{ij}$ and if $\eta_c = 0$ in Equation (3), for an incompressible fluid $\partial\eta/\partial III = 0$. Since to date no fluid has been established to have $\eta_c \neq 0$ (*21*) and since Equation (10) may be represented by a potential (*18*), it is consistent to neglect the effect of III; but this does not imply that the stress must be given by a potential for these fluids. Furthermore in the experiments discussed above it was estimated that III varied by more than 10^6 (*18*); that the data could be correlated without taking the variation of III into account implies that the effect of III is negligibly small.

2c. Since Equation (10) is valid over no more than a limited range in stress, it is important that the stresses encountered in the sphere experiment are the same as those for which the parameters m_1 and n are evaluated. They have been previously estimated to be nearly identical (*18*).

3. The accuracy of the first approximation to the drag coefficient in an Ostwald-de Waele fluid is open to question on two points.

3a. The convergent series for f is truncated after three terms. This may be justified by analogy with the zeroth approximation; the fourth and fifth terms in Equation (22) are relatively small.

3b. The stream function is estimated in terms of only one undetermined parameter a_1. The error involved here is assumed to be small since X_1 for the first approximation is not greatly different from X_0 for the zeroth approximation in Figure 1 for $n > 0.76$ (less than 15% difference).

No definite statement can be made regarding the absolute accuracy of the developments of this paper's second section. However since the correct solution to the equation of motion minimizes J [Equation (6)] or VF_d [Equation (20)], the true plot of X as a function of n lies below all of those shown in Figure 1. This means that the first approximation for Ostwald-de Waele fluid is the best available representation of f for $n > 0.76$ and the corrected result of Tomita is best for $n < 0.76$.

CONCLUSIONS

1. A first approximation to the drag force on a sphere moving slowly through an Ostwald-de Waele fluid is obtained by means of a variational principle proposed by Pawlowski. This is the best available description of the drag force for $n > 0.76$; the corrected zeroth approximation of Tomita is superior for $n < 0.76$. The results of these developments form upper bounds for the drag coefficient as a function of n.

2. Previously available experimental data are reanalyzed and compared with the above developments. For sixty-four points representing three fluids with $n > 0.76$, the average error was 12.0% with the first approximation for Ostwald-de Waele fluid. The corrected zeroth approximation of Tomita predicted sixty-eight points representing two fluids having $n < 0.76$ with an average error of 21.7%.

3. A zeroth approximation is given for a Sisko fluid. No experimental data for fluids described by this model have been reported.

4. The ease with which a fair approximate answer has been obtained for an otherwise extremely difficult problem again demonstrates the usefulness of variational methods.

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NOTATION

a_1 = coefficient depending upon n in Equation (26)
C_i = coefficient in trial expression for stream function ψ, Equation (15)
d_{ij} = rate-of-deformation tensor
D = diameter of capillary tube
f_i = external force vector in equation of motion, Equation (2)
f = drag coefficient defined by Equation (19)
F_d = drag force on sphere (magnitude of vector)
F = integrand of J, Equation (6)
G_i = function of x in trial expression for stream function ψ, Equation (15)
H = term in Equation (17)
H_0 = H for zeroth approximation, Equation (18)
H_1 = H for first approximation, Equation (26)
J = variational integral to be extremized, Equation (6)
L = length of capillary tube
m_0, m_1, n = parameters of Sisko model Equation (8); latter two are also used in Ostwald-de Waele model Equation (10)
N_{R0} = Reynolds number defined by Equation (24)
N_{R1} = Reynolds number defined by Equation (21)
p = pressure, also used for $(n+1)/2$ in Appendix
Q = volume flow rate through capillary tube
r = radial spherical coordinate
R = radius of sphere
t = time
u_i = velocity vector
u_r, u_θ = physical components of the velocity vector in spherical coordinates (9, p. 304)
V = terminal speed of sphere, also may indicate volume of the system as in Equation (6)
X = correction to Stokes's law, Equation (20)
X_0 = value of X for zeroth approximation from Equation (22)
X_1 = value of X for first approximation from Figure 1
x = r/R

Greek Letters

α, φ_0, φ_1 = parameters of Ellis model, Equation (30)
β, η, η_c = variable coefficients in Equation (3)
δ_i^j = Kronecker delta which takes values 0, if i and j are not equal, and 1, if i equals j
ρ = density
ψ = stream function, Equation (14)
θ, φ = spherical coordinates; the angle θ is measured in the x-y plane; φ is measured from the positive z axis
τ_{ij} = viscous portion of the stress tensor

Special Symbols

I, II, III = invariants of the rate-of-deformation tensor defined by Equation (4)
II_τ = second invariant of the tensor τ_{ij} defined by analogy with Equation (4)

Subscripts

$,i$ = covariant differentiation with respect to i^{th} independent variable

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APPENDIX

Rearrangement of Improper Integrals of Second Section into Convergent Series

One may expand $(1 + H)^p$ in a convergent binomial series for $H^2 \leq 1$ and $p = (n+1)/2 > 0$:

$$(1+H)^p = \sum_{r=0}^{\infty} \frac{p!}{(p-r)!\,r!} H^r \tag{1A}$$

This means that (*1*, p. 400)

$$\int_0^\pi \sum_{r=0}^{\infty} \frac{p!}{(p-r)!\,r!} H^r \sin\theta \, d\theta = \sum_{r=0}^{\infty} \int_0^\pi \frac{p!}{(p-r)!\,r!} H^r \sin\theta \, d\theta \tag{2A}$$

Since the right side of Equation (1A) is a convergent series, by the same argument as above

$$\int_1^b \left[\sum_{r=0}^{\infty} \int_0^\pi \frac{p!}{(p-r)!\,r!} H^r \sin\theta \, d\theta \right] x^{-2n} \, dx = \sum_{r=0}^{\infty} \int_1^b \int_0^\pi \frac{p!}{(p-r)!\,r!} H^r \sin\theta \, x^{-2n} \, d\theta \, dx \tag{3A}$$

Observe that for $x \geq 1$

$$\begin{aligned}\int_0^\pi (1+H)^p x^{-2n} \sin\theta \, d\theta &= \int_0^\pi \sum_{r=0}^{\infty} \frac{p!}{(p-r)!\,r!} H^r \sin\theta \, x^{-2n} \, d\theta \\ &\leq \int_0^\pi \sum_{r=0}^{\infty} \left| \frac{p!}{(p-r)!\,r!} H^r \right| \sin\theta \, x^{-2n} \, d\theta \\ &< \int_0^\pi (2)^{p+1} \sin\theta \, x^{-2n} \, d\theta = (2)^{p+2} x^{-2n}\end{aligned} \tag{4A}$$

and that $\int_1^\infty (2)^{p+2} x^{-2n} \, dx$ converges for $n > \frac{1}{2}$. By the Weirstrass M test (*1*, p. 438) the similar improper integrals of the first three terms on the left of Equation (4A) converge as well. This result and Equations (2A) and (3A) may be used to conclude (*1*, p. 451).

$$\begin{aligned}\int_1^\infty \int_0^\pi (1+H)^p \sin\theta \, x^{-2n} \, d\theta \, dx &= \int_1^\infty \sum_{r=0}^{\infty} \int_0^\pi \frac{p!}{(p-r)!\,r!} H^r \sin\theta \, x^{-2n} \, d\theta \, dx \\ &= \sum_{r=0}^{\infty} \int_1^\infty \int_0^\pi \frac{p!}{(p-r)!\,r!} H^r \sin\theta \, x^{-2n} \, d\theta \, dx\end{aligned} \tag{5A}$$